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Hankel hyperdeterminants and Selberg integrals

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Abstract
We investigate the simplest class of hyperdeterminants defined by Cayley in the case of Hankel hypermatrices (tensors of the form $A_{i_1 i_2 \ldots i_k} = f(i_1 + i_2 + \cdots + i_k)$). It is found that many classical properties of Hankel determinants can be generalized, and a connection with Selberg type integrals is established. In particular, Selberg's original formula amounts to the evaluation of all Hankel hyperdeterminants built from the moments of the Jacobi polynomials. Many higher dimensional analogues of classical Hankel determinants are evaluated in closed form. The Toeplitz case is also briefly discussed. In physical terms, both cases are related to the partition functions of one-dimensional Coulomb systems with logarithmic potential.

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1. Introduction

Although determinants have been in use since the mid-eighteenth century, it took almost one hundred years before the modern notation as square arrays was introduced by Cayley [5]. Then, it was not long before Cayley raised the question of extending the notion of determinant to higher dimensional arrays (e.g., cubic matrices A_{ijk}), and proposed several answers, under the name *hyperdeterminants* [6, 7].

The most sophisticated notion of hyperdeterminant has been the object of recent investigations, summarized in [11]. However, the simplest possible generalization of the determinant, defined for a kth order tensor on an n-dimensional space by the k-tuple alternating sum (which vanishes for odd k)

$$\mathrm{Det}_k(A) = \frac{1}{n!} \sum_{\sigma_1,\ldots,\sigma_k \in \mathfrak{S}_n} \epsilon(\sigma_1) \cdots \epsilon(\sigma_k) \prod_{i=1}^{n} A_{\sigma_1(i)\cdots\sigma_k(i)} \tag{1}$$

has been almost forgotten. After the book by Sokolov [36] which contains an exhaustive list of references up to 1960, we have found only [4, 12, 13, 37]. These references contain evaluations of a few higher dimensional analogues of some classical determinants (Vandermonde, Smith, ...). However, the analogues of Hankel determinants do not seem to have been investigated.

In this paper, we shall compute the hyperdeterminantal analogues of various classical Hankel determinants. The elements of these determinants will in general be combinatorial numbers or orthogonal polynomials. Our main technique will be the use of integral representations. We shall see that the relevant tool is Selberg's integral and its generalizations, mainly in the form given by Kaneko. More than this, we can say that the knowledge embodied in Selberg's formula and its limiting cases amounts to a closed form evaluation of all Hankel hyperdeterminants built from the moment sequences associated with the classical orthogonal polynomials.

A more general class of hyperdeterminants is given by the partition functions $Z_n(\beta)$ of log-potential Coulomb systems, when β is an even integer. When the particle–background interaction does not lead to a Selberg integral, the partition function can usually be evaluated in a more or less closed form only for $\beta = 1, 2, 4$ (as a Pfaffian or a determinant). The evaluation of general hyperdeterminants is of course much more difficult, but their simple transformation properties leave some hope that at least some of them may be evaluated by higher dimensional analogues of the algebraic techniques working for Hankel determinants.

The consideration of the partition functions of similar Coulomb systems with the particles confined on a circle suggests immediately the following definition. A tensor $T_{i_1\ldots i_k}^{j_1\ldots j_k}$ will be called a *Toeplitz tensor* if

$$T_{i_1\ldots i_k}^{j_1\ldots j_k} = f(i_1 + \cdots + i_k - j_1 - \cdots - j_k). \tag{2}$$

Indeed, when β is an even integer, the partition function turns out to be a Toeplitz hyperdeterminant.

2. Hankel hyperdeterminants

Let $\left(A_{i_1\ldots i_k}\right)_{0\leqslant i_1,\ldots,i_k\leqslant n-1}$ be a tensor of order k and dimension n. The tensor A is said to be a Hankel tensor if $A_{i_1\cdots i_k} = f(i_1 + \cdots + i_k)$.

Let us now fix some sequence $c = (c_n)_{n\geqslant 0}$, and consider the hyperdeterminants

$$D_n^{(k)}(c) = \mathrm{Det}_{2k}\left(c_{i_1+\cdots+i_{2k}}\right)_{0\leqslant i_p\leqslant n-1} \tag{3}$$

as defined by formula (1), in which $\mathfrak{S}_n$ is the symmetric group and $\epsilon(\sigma)$ the signature of a permutation σ. For $k = 1$, this is an ordinary Hankel determinant. For $n = 2$, it is easy to derive the expression

$$D_2^{(k)}(c) = \frac{1}{2}\sum_{i=0}^{2k}(-1)^i\binom{2k}{i}c_i c_{2k-i} \tag{4}$$

whose right-hand side is well known in classical invariant theory (it is one-half of the apolar covariant of the binary form $f(x,y) = \sum_i \binom{2k}{i} c_i x^i y^{2k-i}$ with itself, see [21]).

The case $c_n = n!$ will be used as a running example throughout this paper. Using (4), we can give our first illustration of a higher order determinant

$$D_2^{(k)}(c) = \frac{1}{2}\sum_{i=0}^{2k}(-1)^i(2k)! = \frac{1}{2}(2k)! \tag{5}$$

which will provide a check for the general case.

Let now μ be the linear functional on the space of polynomials in one variable such that $\mu(x^n) = c_n$. We extend it to polynomials in several variables by setting $\mu_n\left(x_1^{m_1}\cdots x_n^{m_n}\right) = c_{m_1}\cdots c_{m_n}$. Then, using the expansion of the Vandermonde determinant

$$\Delta(x) = \prod_{i>j}(x_i - x_j) = \sum_{\sigma\in\mathfrak{S}_n}\epsilon(\sigma)\sigma\left(x_n^{n-1}x_{n-1}^{n-2}\cdots x_2\right) \tag{6}$$

where a permutation σ acts on a monomial by sending each x_i on $x_{\sigma(i)}$, it is easily seen that

$$D_n^{(k)}(c) = \frac{1}{n!}\mu_n(\Delta^{2k}(x)). \tag{7}$$

Expanding each factor $(x_i - x_j)^{2k}$ by the binomial theorem, we obtain

$$D_n^{(k)}(c) = \frac{1}{n!}\sum_{M=(m_{ij})}(-1)^{|M|}\prod_{i>j}\binom{2k}{m_{ij}}\prod_{p=1}^{n}c_{\alpha_p(M)} \tag{8}$$

where M runs over all strictly lower triangular integer matrices such that $0 \leqslant m_{ij} \leqslant 2k$, $|M| = \sum_{i>j} m_{ij}$, and

$$\alpha_p(M) = 2k(p-1) + \sum_{i=p+1}^{n} m_{ip} - \sum_{j=1}^{p-1} m_{pj}. \tag{9}$$

This extends (4) and provides a faster algorithm than the definition.

Now, if μ is a measure on the real line, then

$$D_n^{(k)}(c) = \frac{1}{n!}\int_{\mathbb{R}^n}\Delta^{2k}(x)\,\mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_n). \tag{10}$$

When $k=1$, this is a well-known formula due to Heine [14]. For arbitrary k, the integral can be evaluated in closed form in many interesting cases by means of Selberg's integral formula [34] which gives, for

$$S_n(a,b,k) = \int_0^1\cdots\int_0^1 |\Delta(x)|^{2k}\prod_{i=1}^{n}x_i^{a-1}(1-x_i)^{b-1}\,\mathrm{d}x_i \tag{11}$$

the value

$$S_n(a,b,k) = \prod_{j=0}^{n-1}\frac{\Gamma(a+jk)\Gamma(b+jk)\Gamma((j+1)k+1)}{\Gamma(a+b+(n+j-1)k)\Gamma(k+1)}. \tag{12}$$

The formula is valid, when defined, for complex values of k as well, but it is interesting to observe that most of its known proofs start with the assumption that k is a positive integer, and then extend the result by means of Carlson's theorem (see, e.g., [30]).

Hence, Selberg's integral computes precisely the Hankel hyperdeterminants $D_n^{(k)}(c)$ when the c_n are the moments of the measure $\mathrm{d}\mu(x) = \mathbf{1}_{[0,1]}x^{a-1}(1-x)^{b-1}\,\mathrm{d}x$ (the Beta distribution). The orthogonal polynomials for this measure are, up to a simple change of variables, the Jacobi polynomials $P_n^{(a-1,b-1)}(1-2x)$. Hence, we can as well compute $D_n^{(k)}(c)$ for the moments of the ordinary Jacobi polynomials, and their limiting cases (Laguerre and Hermite). The appropriate variants of Selberg's formula are listed in [30] (17.6.)

When the measure $\mathrm{d}\mu(x)$ does not lead to a known variant of Selberg's formula (such as Aomoto's and Kaneko's generalizations, which are discussed below), it is sometimes possible to evaluate the hyperdeterminant of order 4 ($k=2$) from the knowledge of the scalar products $\langle P_n, P_m'\rangle$ of the corresponding (monic) orthogonal polynomials and their derivatives. Indeed, it is classical (see [29]) that

$$\Delta(x)^4 = \det(P_{i-1}(x_j)|P_{i-1}'(x_j)) \tag{13}$$

where, in the right-hand side, we mean the $2n\times 2n$-matrix with $i=1,\ldots,2n$, $j=1,\ldots,n$, whose first n columns are the P_i and the last n ones their derivatives. Using one of de Bruijn's formulae (see [30], A.18.7), we can write the hyperdeterminant as a Pfaffian

$$\mathrm{Det}_4(c_{i+j+k+l})|_0^{n-1} = \mathrm{pf}(M_{ij})|_0^{2n-1} \tag{14}$$

where M is the skew-symmetric matrix such that

$$M_{ij} = \langle P_i, P_j' \rangle = \int_a^b P_i(x) P_j'(x) \, \mathrm{d}\mu(x) \tag{15}$$

if $i < j$. For example, if $\mathrm{d}\mu(x) = \mathrm{e}^{-x}\, \mathrm{d}x$ is the Laguerre measure on $[0, \infty)$, whose monic orthogonal polynomials are given by the generating series

$$\sum_{n \geqslant 0} \tilde{L}_n(x) \frac{(-t)^n}{n!} = \frac{\mathrm{e}^{\frac{xt}{t-1}}}{1-t} \tag{16}$$

an easy calculation gives

$$\sum_{n,m \geqslant 0} \langle \tilde{L}_n(x), \tilde{L}_m'(x) \rangle \frac{(-1)^{m+n}}{n!m!} t^n s^m = \frac{s}{(1-s)(1-st)} \tag{17}$$

which leads to

$$\mathrm{Det}_4((i+j+k+l)!)|_0^{n-1} = \mathrm{pf}((-1)^{i+j-\delta_{i>j}} i! j!)|_0^{2n-1} \tag{18}$$

with $\delta_{i>j} = 1$ if $i > j$ and 0 otherwise. The calculation of the Pfaffian is straightforward, and we obtain

$$\mathrm{Det}_4((i+j+k+l)!)|_0^{n-1} = \prod_{i=0}^{2n-1} i! \tag{19}$$

a special case of the general formula (29) derived below from Selberg's integral.

For later reference, let us recall that the Hankel determinants $D_n^{(1)}(c)$ are the products of the squared norms of the monic orthogonal polynomials P_n, and that, more generally, the shifted Hankel determinants $D_{n;r}^{(1)}(c) = D_n^{(1)}(c^{(r)})$, associated with the shifted sequences $c_n^{(r)} = c_{n+r}$ are given by

$$D_{n;r}^{(1)}(c) = \det(\langle x^r P_i, P_j \rangle)|_0^{n-1}. \tag{20}$$

Similarly, the shifted hyperdeterminants of order 4 can be reduced to Pfaffians

$$D_{n;r}^{(2)}(c) = \mathrm{pf}((-1)^{\delta_{i<j}} \langle x^r P_i, P_j' \rangle)|_0^{2n-1}. \tag{21}$$

Finally, let us remark that when the measure can be written in the form $\mathrm{d}\mu(x) = C\, \mathrm{e}^{-\lambda V(x)}\, \mathrm{d}x$, the integral (10) represents the partition function of a one-component log-potential Coulomb system on the line, evaluated at $\beta = 2k$ (see, e.g., [10]). It is a common feature of most of these systems that the partition function may be evaluated in closed form only for $\beta = 1, 2, 4$ (the case $\beta = 1$ can be formulated in terms of Pfaffians of bimoments of skew-orthogonal polynomials, see [29], and will not concern us here).

Similarly, the partition function of a one-component Coulomb system of n identical particles confined on the unit circle has the general form

$$Z_n(\beta) = C_n \frac{1}{(2\pi \mathrm{i})^n} \oint \frac{\mathrm{d}z_1}{z_1} \cdots \oint \frac{\mathrm{d}z_n}{z_n} |\Delta(z)|^\beta \prod_{j=1}^N \mathrm{e}^{-\beta V(z_j)}. \tag{22}$$

For $\beta = 2k$, this is, up to a scalar factor, the hyperdeterminant of the Toeplitz tensor associated with the bi-infinite sequence

$$c_n = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} z^{-n}\, \mathrm{e}^{-\beta V(z)} \frac{\mathrm{d}z}{z}. \tag{23}$$

As above, the knowledge of the appropriate orthogonal polynomials allows one to evaluate the determinant ($k = 1$) and sometimes the fourfold hyperdeterminant.

3. Examples derived from Selberg's integral

The evaluation of Hankel determinants built on classical sequences of combinatorial numbers arises in many contexts (see [20, 40] and references therein). Recent work on the theory of coherent states has led to the discovery of integral representations of many such sequences [31, 32]. For the sequences proportional to moments of a Beta distribution, the Hankel hyperdeterminants can be evaluated immediately. We have

$$\mathrm{B}(a+n,b) = \mathrm{B}(a,b)\frac{(a)_n}{(a+b)_n} \tag{24}$$

so that it is sufficient to evaluate the hyperdeterminants for the case $c_n = (a)_n/(a+b)_n$, which yields

$$D_n^{(k)}(c) = \frac{1}{n!(k!)^n}\prod_{j=0}^{n-1}\frac{(a)_{jk}(b)_{jk}(kj+k)!}{(a+b)_{(n+j-1)k}}. \tag{25}$$

As a limit case, the Laguerre–Selberg integral (see [30] (17.6.5))

$$LS_n(\alpha,\gamma) = \int_{(0,\infty)^n}|\Delta(x)|^{2\gamma}\prod_{j=1}^{n}x_j^{\alpha-1}\,\mathrm{e}^{-x_j}\,\mathrm{d}x_j = \prod_{j=0}^{n-1}\frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)}{\Gamma(1+\gamma)} \tag{26}$$

gives for the moments

$$c_n = \Gamma(n+\alpha) = \Gamma(\alpha)(\alpha)_n \tag{27}$$

of the Gamma distribution $\mathrm{d}\mu(x) = x^\alpha\,\mathrm{e}^{-x}\,\mathrm{d}x$ the value

$$D_n^{(k)}(c) = \frac{1}{n!k!^n}\prod_{j=0}^{n-1}(k+jk)!\Gamma(\alpha+jk). \tag{28}$$

Here are a few examples extending some known evaluations of Hankel determinants.

3.1. Factorials and Γ functions

For $\alpha = r+1$ a positive integer, (28) gives the shifted hyperdeterminant $D_{n;r}^{(k)}$ of factorials. In particular,

$$\mathrm{F}(n,k) := D_n^{(k)}(c) = \frac{1}{n!k!^n}\prod_{j=0}^{n-1}(k+kj)!(kj)! \tag{29}$$

thus recovering the classical evaluation $D_n^{(1)}(c) = [1!2!\cdots(n-1)!]^2$ (see, e.g., [40]) of the Hankel determinant.

3.2. Catalan numbers

Here, we take $c_n = C_n = \frac{1}{n+1}\binom{2n}{n}$. As pointed out in [31],

$$C_n = \frac{1}{2\pi}\int_0^4 x^n\sqrt{\frac{4-x}{x}}\,\mathrm{d}x = \frac{2^{2n+1}}{\pi}\mathrm{B}\left(n+\frac{1}{2},\frac{3}{2}\right) \tag{30}$$

so that

$$\begin{aligned} D_n^{(k)}(c) &= \frac{2^{2kn(n-1)+n}}{n!\pi^n}S_n\left(\frac{1}{2},\frac{3}{2},k\right) \\ &= \frac{2^{kn(n-1)-n}}{n!k!^n}\prod_{j=0}^{n-1}\frac{(k+kj)!(2kj+1)!!(2kj-1)!!}{(1+k(n+j-1))!} \end{aligned} \tag{31}$$

where $(2n+1)!! = 1 \cdot 3 \cdots (2n+1)$. The Hankel hyperdeterminants of shifted Catalan numbers, obtained by replacing c_n by c_{r+n} leads to another Selberg integral

$$D_{n;r}^{(k)}(c) = \frac{2^{2kn(n-1)+n(2r+1)}}{n!\pi^n} S_n\left(r+\frac{1}{2}, \frac{3}{2}, k\right). \tag{32}$$

In the case $k=1$, the Hankel determinants of shifted Catalan numbers have been computed by Desainte-Catherine and Viennot [9] with the aim of enumerating the Young tableaux with even columns and height at most $2n$. It would be interesting to find a combinatorial interpretation of the hyperdeterminants of shifted Catalan numbers, which are still positive integers.

3.3. Central binomial coefficients

Another example of a classical sequence represented by the Beta distribution, mentioned in [32], is

$$\binom{2n}{n} = \frac{1}{\pi}\int_0^4 x^n[x(4-x)]^{-1/2}\,\mathrm{d}x = \frac{4^n}{\pi}\mathrm{B}\left(n+\frac{1}{2}, \frac{1}{2}\right) \tag{33}$$

so that, for $c_n = \binom{2n}{n}$,

$$D_{n;r}^{(k)} = \frac{4^{kn(n-1)+nr}}{n!\pi^n} S_n\left(r+\frac{1}{2}, \frac{1}{2}, k\right). \tag{34}$$

3.4. Other combinatorial sequences

The On-line Encyclopaedia of Integer Sequences [35] provides many further examples of interesting combinatorial sequences arising from the Beta distribution. For example, it is observed in [32] that the sequence `A001813` has the representation

$$\frac{(2n)!}{n!} = \frac{1}{2\sqrt{\pi}}\int_0^\infty x^n\,\mathrm{e}^{-x/4}x^{-1/2}\,\mathrm{d}x \tag{35}$$

which yields to

$$D_{n;r}^{(k)}(c) = \pi^{-\frac{n}{2}}4^{n[k(n-1)+r]}LS_n\left(r+\tfrac{1}{2}, k\right). \tag{36}$$

Less familiar examples, pointed out by K Penson (personal communication), are the sequences `A004981` and `A004987`.

3.5. Hilbert hyperdeterminants

Another classical example of a Hankel determinant which can be evaluated in closed form is the Hilbert determinant

$$\left|\frac{1}{i+j-1}\right|_{i,j=1}^{n} = D_n^{(1)}(c) \tag{37}$$

where

$$c_n = \int_0^1 x^n\,\mathrm{d}x. \tag{38}$$

Here, (10) gives immediately

$$\mathrm{H}(k,n) := D_n^{(k)}(c) = \frac{1}{n!}S_n(1,1,k). \tag{39}$$

Setting $c_n(a) = (n+a+1)^{-1}$, we obtain in the same way

$$D_n^{(k)}(c(a)) = \frac{1}{n!}S_n(1+a, 1; k). \tag{40}$$

For $a=r$, we obtain the hyperdeterminants $D_{n;r}^{(k)}(c)$.

4. Discrete measures

As we have seen, the information contained in Selberg's integral amounts to the evaluation of the Hankel hyperdeterminants built from moments of the Beta distribution. As special cases, one finds many classical combinatorial sequences, of which the Hankel determinants have been considered in the literature. It is therefore natural, when looking for extensions of Selberg's integral, to try to evaluate in closed form the Hankel hyperdeterminants of the simplest combinatorial sequences not arising from the Beta distribution. Among them, one finds several sequences which can be represented as moments of a discrete measure. We shall briefly describe a couple of examples related to two of the simplest families of orthogonal polynomials of a discrete variable (Charlier and Krawtchouk). Although we have not been able to find a closed form for the hyperdeterminants, we obtained on the way a couple of apparently new formulae for some Hankel determinants. More interesting identities involving these (and other) orthogonal polynomials will be obtained in section 5.

4.1. The Poisson distribution: Bell numbers and polynomials

We now take $c_n = b_n(a)$, the (one-variable) Bell polynomials, defined by

$$b_0(a) = 1 \qquad \text{and} \qquad b_n(a) = \sum_{k=1}^{n} S(n,k)a^k \tag{41}$$

where the $S(n,k)$ are the Stirling numbers of the second kind (so that $b_n(1)$ are the Bell numbers). These are the moments of the discrete measure (the Poisson distribution)

$$\mathrm{d}\mu_a(x) = \mathrm{e}^{-a} \sum_{k\geqslant 0} \frac{a^k}{k!}\delta(x-k) \tag{42}$$

for which the Charlier polynomials are the orthogonal system (cf [18]). The monic Charlier polynomials $C_n^{(a)}(x)$ satisfy

$$\left\langle C_n^{(a)}, C_n^{(a)} \right\rangle = a^n n! \tag{43}$$

whence the classical evaluation of the Hankel determinants [38]

$$D_n^{(1)} = a^{n(n-1)/2} \prod_{j=0}^{n-1} j!. \tag{44}$$

No analogue of Selberg's integral is known for the measure $\mathrm{d}\mu_a$, and finding a closed form for the hyperdeterminants would amount to finding such a generalization. For now, the best that we can do is to evaluate the fourth-order hyperdeterminants by means of formula (14). With this aim, we need the scalar products $\left\langle C_n^{(a)}, C_m^{(a)\prime} \right\rangle$, which can be easily obtained from the generating function

$$C(u,x;a) = \sum_{n\geqslant 0} C_n^{(a)}(x)\frac{u^n}{n!} = \mathrm{e}^{-au}(1+u)^x. \tag{45}$$

Taking the scalar product of this expression with $\frac{\partial C(v,x;a)}{\partial x}$, we find that

$$\left\langle C_n^{(a)}, C_m^{(a)\prime} \right\rangle = \begin{cases} (-1)^{m-n+1}\frac{a^n m!}{m-n} & \text{if} \quad m > n \\ 0 & \text{otherwise.} \end{cases} \tag{46}$$

It would remain to find a closed expression for the Pfaffian (14). The first values are

$$D_2^{(2)}(c) = a(1+6a)$$
$$D_3^{(2)}(c) = 8a^3(1+24a+45a^2+90a^3)$$
$$D_4^{(2)}(c) = 1728a^6(1+60a+360a^2+2080a^3+2415a^4+2100a^5+2100a^6).$$

It is interesting to observe that the shifted determinants $D_{n;r}^{(1)}(c)$ can be expressed as Wronskians

$$D_{n;r}^{(1)}(c) = a^{n(n-1)/2}W(b_r, b_{r+1}, \ldots, b_{r+n-1})(a). \tag{47}$$

This identity follows immediately from the recursion

$$b_{n+1}(a) = a[b_n(a) + b'_n(a)] \tag{48}$$

and is not of the same nature as the Karlin–Szegö-type identities such as (53) below, discussed in section 5, in which the shifted Hankel determinant of order n is expressed in terms of a Wronskian of order r. Here, also, $D_{n;r}^{(1)}(c)$ is always divisible by $D_n^{(1)}(c)$. In this case, the explanation is simple: it follows from (20) that

$$D_{n;r}^{(1)}(c) = \det(\langle x^r P_i, P_j^*\rangle) D_n^{(1)}(c) \tag{49}$$

where P_j^* is the adjoint basis of P_i. The ratio $D_{n;r}^{(1)}(c)\big/D_n^{(1)}(c)$ is, therefore, the determinant of the operator $X_n^{(r)} = M_r \circ \pi_n$ where M_r is multiplication by x^r and π_n the orthogonal projection on the subspace spanned by $P_0, \ldots, P_{n-1}$. The matrix elements of $X_n^{(r)}$ can be read directly from the three-term recurrence relation of the monic polynomials, by iterating it, if necessary, to express $x^r P_i$ as a linear combination of the P_j. The matrix element $\langle x^r P_i, P_j^*\rangle$ is then equal to the coefficient of P_j in this expression.

For example, if the $P_n = C_n^{(a)}(x)$ are the monic Charlier polynomials, the three-term recurrence is

$$xP_i = P_{i+1} + (i+a)P_i + iaP_{i-1} \tag{50}$$

so that

$$x^2P_i = P_{i+2} + (2i+1+2a)P_{i+1} + (a^2+a+4ai+i^2)P_i + ia(2i-1+2a)P_{i-1} + i(i-1)aP_{i-2} \tag{51}$$

and for $r = 2$ and $n = 3$, the matrix is

$$X_3^{(2)} = \begin{bmatrix} a+a^2 & a+2a^2 & 2a^2 \\ 1+2a & 1+5a+a^2 & 6a+4a^2 \\ 1 & 3+2a & 4+9a+a^2 \end{bmatrix} \tag{52}$$

whose determinant is $6a^3 + 6a^4 + 3a^5 + a^6$. This is the value at $x = 0$ of the 2-Wronskian $W\big(C_3^{(a)}, C_4^{(a)}\big)(0)$ of Charlier polynomials. There is a general formula (apparently new)

$$D_{n;r}^{(1)}(c) = (-1)^{rn}\frac{W\big(C_n^{(a)}, \ldots, C_{n+r-1}^{(a)}\big)(0)}{1!2!\cdots(r-1)!}D_n^{(1)}(c) \tag{53}$$

which will be derived in section 5.

4.2. The binomial distribution and a class of Hankel–Wronskians

Hankel determinants built on the moments of the binomial distribution arise for example in [40] (4.12.3), where one finds a discussion of the Hankel determinants $D_n^{(1)}(c)$ associated with the sequence

$$c_n = \frac{\mathrm{d}^n}{\mathrm{d}t^n}f(t) \qquad \text{with} \qquad f(t) = \left(\frac{\mathrm{e}^t}{1-\mathrm{e}^t}\right)^x \tag{54}$$

whose particular case $x = 1$ gives back one of the determinants computed by Lawden [25]. To investigate the hyperdeterminantal analogues, we will find it convenient to make the substitution $t \to \mathrm{i}\pi - t$, and to assume at first that $-x = N$ is a positive integer. Up to a trivial sign, we can now take $f(t) = (1 + \mathrm{e}^t)^N$, and our sequence is

$$\begin{aligned} c_n &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}(1+\mathrm{e}^t)^N \\ &= \sum_{k=0}^{N} k^n \binom{N}{k} \mathrm{e}^{kt} \\ &= (1+\mathrm{e}^t)^N \sum_{k=0}^{N} k^n \binom{N}{k} \left(\frac{\mathrm{e}^t}{1+\mathrm{e}^t}\right)^k \left(1 - \frac{\mathrm{e}^t}{1+\mathrm{e}^t}\right)^{N-k} \\ &= (1-p)^{-N} \sum_{k=0}^{N} k^n \binom{N}{k} p^k (1-p)^{N-k} \end{aligned} \tag{55}$$

where $p = \mathrm{e}^t/(1+\mathrm{e}^t)$. That is, (c_n) is the moment sequence of the binomial distribution, for which the Krawtchouk polynomials $K_n(x; p, N)$ are orthogonal. We have

$$\sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} K_m(k) K_n(k) = \frac{(-1)^n n!}{(-N)_n} \left(\frac{1-p}{p}\right)^n \delta_{mn} \tag{56}$$

whilst the monic polynomials $\tilde{K}_n$ are related to the standard ones by

$$\tilde{K}_n(x) = p^n (-N)_n K_n(x). \tag{57}$$

Hence,

$$\|\tilde{K}_n\|^2 = (-1)^n n! p^n (1-p)^n (-N)_n \tag{58}$$

which gives for the Hankel determinant

$$\begin{aligned} D_n^{(1)}(c) &= (1-p)^{-Nn} \prod_{j=0}^{n-1} \|\tilde{K}_j\|^2 \\ &= \frac{(-\mathrm{e}^t)^{n(n-1)/2}}{(1+\mathrm{e}^t)^{n(-N+n-1)}} \prod_{j=0}^{n-1} j!(-N)_j \end{aligned} \tag{59}$$

which agrees with theorem 4.59 of [40] after substituting back $t \to \mathrm{i}\pi - t$ and $N = -x$, and can be extended as usual to values of x ranging over the whole complex plane by means of Carlson's theorem.

Although we could not find a closed form for the hyperdeterminants, we may remark that the interpretation in terms of the Krawtchouk polynomials allows one to go one step further with the determinants, and to find a closed form for the $D_{n;1}^{(1)}$, which seems to be a new result.

Indeed, we know that

$$D_{n;1}^{(1)} = \det(X_n) D_n^{(1)} \tag{60}$$

where X_n is the operator of multiplication by x followed by the orthogonal projection on the subspace spanned by the first n Krawtchouk polynomials. The matrix of X_n can be read directly on the three-term recurrence

$$x\tilde{K}_n(x) = \tilde{K}_{n+1}(x) + [p(N-n) + n(1-p)]\tilde{K}_n(x) + np(1-p)(N+1-n)\tilde{K}_{n-1}(x) \tag{61}$$

which yields the tridiagonal matrix

$$X_n = \begin{bmatrix} \beta & 1 & 0 & 0 & \cdots & 0 & 0 \\ \lambda(\mu-1) & \alpha+\beta & 1 & 0 & \cdots & 0 & 0 \\ 0 & 2\lambda(\mu-2) & 2\alpha+\beta & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & & & \vdots \\ 0 & 0 & 0 & & \cdots & (n-1)\lambda(\mu-n+1) & (n-1)\alpha+\beta \end{bmatrix} \tag{62}$$

where $\alpha = 1 - 2p, \beta = Np, \lambda = p(1-p)$ and $\mu = N + 1$. The three-term recurrence for the tridiagonal determinants is easily solved by means of a generating function, and one finds

$$\det(X_n) = (N)_n p^n. \tag{63}$$

The other shifted determinants $D^{(1)}_{n;r}$ can in principle be calculated by the same method, but it does not seem possible to solve the recurrences in closed form, and indeed, numerical calculations show that no nice factorized expression can be expected, except in the special case $r = 2$ and $N = -1$, which gives back another one of Lawden's determinants [25]. For the operator $X^{(2)}_n$, multiplication by x^2 followed by projection, we obtain, for $N = -1$,

$$\det\big(X^{(2)}_n\big) = (n!)^2\, \mathrm{e}^{nt}\, \frac{\mathrm{e}^{(n+1)t} - (-1)^{n+1}}{(1+\mathrm{e}^t)^{2n+1}} \tag{64}$$

but no other case seems to lead to an interesting formula.

5. Examples involving orthogonal polynomials

Hankel determinants associated with sequences of the form $c_n = Q_n(x)$, where (Q_n) is a family of orthogonal polynomials, have been called *Turánians* by Karlin and Szegö, who computed their values for the classical families [17]. Recent references on this subject can be found in [26], where these results have been generalized by a different method based on a little-known determinantal identity due to Turnbull.

In this section, we will calculate the hyperdeterminantal analogues of the Turánians evaluated in [17]. As in the preceding section, we will make use of the integral representations of the classical orthogonal polynomials. Interestingly enough, Selberg's formula will not be sufficient to deal with these cases, and we will have to rely upon one of its extensions, which is due to Kaneko [15].

5.1. Kaneko's integral and its variants

The required integral formula involves the *generalized Jacobi polynomials* $p^{\alpha,\beta,\gamma}_\kappa(y)$ [8, 19, 22, 41], which are the symmetric polynomials in r variables $(y_1, \ldots, y_r)$ obtained by applying the Gram–Schmidt process to the basis of monomial symmetric functions $m_\mu(y)$ (ordered by the condition $\mu < \nu$ if $|\mu| < |\nu|$, or $|\mu| = |\nu|$ and μ precedes ν for the reverse lexicographic order) with respect to the measure

$$\mathrm{d}\mu^{\alpha,\beta,\gamma}(y) = |\Delta(y)|^{2\gamma+1} \prod_{i=1}^{r} (1-y_i)^\alpha (1+y_i)^\beta\, \mathrm{d}y_1 \cdots \mathrm{d}y_r \tag{65}$$

on $[-1, 1]^r$, normalized by the condition that the leading term of $p^{\alpha,\beta,\gamma}_\kappa(y)$ is $m_\kappa(y)$. Let

$$R(x, y) = \prod_{i=1}^{n} \prod_{j=1}^{r} (x_i - y_j). \tag{66}$$

Kaneko's formula reads

$$\int_{[0,1]^n} R(x,y)\prod_{i=1}^n x_i^{a-1}(1-x_i)^{b-1}|\Delta(x)|^{2c}\,\mathrm{d}x_1\cdots\mathrm{d}x_n$$
$$= 2^{-nr}S_n(a,b;c)\,p_{(n^r)}^{\alpha,\beta,\gamma}(1-2y_1,\ldots,1-2y_r) \tag{67}$$

where $\alpha=\frac{a}{c}-1$, $\beta=\frac{b}{c}-1$ and $\gamma=c-\frac{1}{2}$.

The multivariate Jacobi polynomials indexed by rectangular partitions can be expressed in terms of generalized hypergeometric functions. This expression is simpler in terms of the polynomials

$$P_\kappa^{(a,b)}\left(y_1,\ldots,y_r;\frac{1}{c}\right)=\frac{p_\kappa^{a,b,c-\frac{1}{2}}(1-2y_1,\ldots,1-2y_r)}{p_\kappa^{a,b,c-\frac{1}{2}}(1,\ldots,1)} \tag{68}$$

which are orthogonal on $[0,1]^r$ for the Selberg measure with parameters $(a+1,b+1,c)$. For a rectangular partition $\kappa=(n^r)$,

$$P_{(n^r)}^{(a,b)}\left(y_1,\ldots,y_r;\frac{1}{c}\right)={}_2F_1^{(1/c)}\left(\begin{matrix}-n,a+b+s+n\\ a+s\end{matrix}\middle| y_1,\ldots,y_r\right) \tag{69}$$

where $s=1+(r-1)c$. The generalized hypergeometric functions associated with Jack polynomials $C_\kappa^{(\alpha)}(y_1,\ldots,y_r)$ are defined by [15, 19]

$${}_pF_q^{(\alpha)}\left(\begin{matrix}a_1\ldots a_p\\ b_1\ldots b_q\end{matrix}\middle| y_1,\ldots,y_r\right)=\sum_{n\geqslant 0}\frac{1}{n!}\sum_{|\kappa|=n}\frac{[a_1]_\kappa^{(\alpha)}\cdots[a_p]_\kappa^{(\alpha)}}{[b_1]_\kappa^{(\alpha)}\cdots[b_q]_\kappa^{(\alpha)}}C_\kappa^{(\alpha)}(y_1,\ldots,y_r) \tag{70}$$

where

$$[a]_\kappa^{(\alpha)}=\prod_{i=1}^{\ell(\kappa)}\left(a-\frac{1}{\alpha}(i-1)\right)_{\kappa_i}. \tag{71}$$

We note that in the case $\kappa=(n^r)$, the denominator of (68) is given by (67) as

$$p_\kappa^{a,b,c-\frac{1}{2}}(1,\ldots,1)=2^{nr}\frac{S_n(a,b+r,c)}{S_n(a,b,c)}. \tag{72}$$

This formula is needed to calculate the degenerate cases of Kaneko's integral corresponding to Laguerre and Hermite polynomials.

The generalized Laguerre polynomials $L_\kappa^a(y;\alpha)$ are defined by [23] (see also [2])

$$L_\kappa^a(y_1,\ldots,y_r;\alpha)=\lim_{b\to\infty}P_\kappa^{(a,b)}\left(\tfrac{y_1}{b},\ldots,\tfrac{y_r}{b};\alpha\right) \tag{73}$$

(we use there the convention of [42]). Let $LS_n(a,c)$ denote the Laguerre–Selberg integral (26). One can deduce from (67) the Laguerre version of Kaneko's integral. Indeed, Kaneko's formula can also be written as [15]

$$\int_{[0,1]^n} R(x,y)|\Delta(x)|^{2c}\prod_{i=1}^n x_i^{a-1}(1-x_i)^{b-1}\,\mathrm{d}x_i$$
$$=S_n(a+r,b,c)\,{}_2F_1^{(c)}\left(\begin{matrix}-n,\frac{1}{c}(a+b+r-1)+n-1\\ \frac{1}{c}(a+r)-1\end{matrix}\middle| y_1,\ldots,y_r\right). \tag{74}$$

Setting $x_i = u_i/L$, $y_i = v_i/L$ and letting $L \to \infty$ in this formula we obtain

$$\int_{(0,\infty)^n} R(u, v)|\Delta(u)|^{2c} \prod_{i=1}^{n} u_i^{a-1}\,\mathrm{e}^{-u_i}\,\mathrm{d}u_i = \lim_{L\to\infty} L^{nr+cn(n-1)+n(a-1)+n} S_n(a+r, L+1, c) \times {}_2F_1^{(c)}\left(\begin{matrix} -n, \frac{1}{c}(a+r+L)+n-1 \\ \frac{1}{c}(a+r-1)\end{matrix}\middle| \frac{v_1}{L}, \ldots, \frac{v_r}{L}\right). \tag{75}$$

From (69) and (73) we have, setting $b' = \frac{L+1}{c} - 1$,

$$\begin{aligned}\lim_{L\to\infty} {}_2F_1^{(c)}&\left(\begin{matrix} -n, \frac{1}{c}(a+r+L)+n-1 \\ \frac{1}{c}(a+r-1)\end{matrix}\middle| \frac{v_1}{L}, \ldots, \frac{v_r}{L}\right) \\ &= \lim_{b'\to\infty} {}_2F_1^{(c)}\left(\begin{matrix} -n, a'+b'+s'+n \\ a'+s'\end{matrix}\middle| \frac{v_1}{cb'+c-1}, \ldots, \frac{v_r}{cb'+c-1}\right) \\ &= L_{n^r}^{a'}\left(\frac{v_1}{c}, \ldots, \frac{v_r}{c}; c\right)\end{aligned} \tag{76}$$

where $a' = \frac{a}{c} - 1$ and $s' = 1 + \frac{r-1}{c}$.

On the other hand,

$$\lim_{L\to\infty} L^{nr+cn(n-1)+n(a-1)+n} S_n(a+r, L+1; c) = LS_n(a+r, c). \tag{77}$$

Finally, we get

$$\int_{(0,\infty)^n} R(x, y)\Delta^{2k}(x) \prod_{i=1}^{n} x_i^{a-1}\,\mathrm{e}^{-x_i}\,\mathrm{d}x_i = LS_n(a+r, c) L_{(n^r)}^{\frac{a}{c}-1}\left(\frac{y_1}{c}, \ldots, \frac{y_r}{c}; c\right). \tag{78}$$

Similarly, an appropriate limit of (67) yields

$$\int_{\mathbb{R}^n} R(x, y)\Delta^{2k}(x) \prod_{i=1}^{n} \mathrm{e}^{-x_i^2}\,\mathrm{d}x_i = (-1)^{\frac{nr}{2}} \pi^{\frac{n}{2}} 2^{-\frac{1}{2}kn(n-1)-nr} k^{\frac{nr}{2}} \prod_{j=1}^{n} \frac{(kj)!}{k!} \times H_{(n^r)}\left(\mathrm{i}\frac{y_1}{\sqrt{k}}, \ldots, \mathrm{i}\frac{y_r}{\sqrt{k}}; k\right) \tag{79}$$

where the generalized Hermite polynomials $H_\kappa(y; \alpha)$ are defined by

$$H_\kappa(y_1, \ldots, y_r; \alpha) = \lim_{a\to\infty} (-\sqrt{2a})^{|\kappa|} L_\kappa^a(a + y_1\sqrt{2a}, \ldots, a + y_r\sqrt{2a}; \alpha). \tag{80}$$

We follow here the convention of [2].

Kaneko's identity can be interpreted as a generalization of Heine's integral representation of orthogonal polynomials in the Jacobi case. Indeed, it can be rewritten as

$$p_{(n^r)}^{\alpha,\beta,\gamma}(t_1, \ldots, t_r) = \frac{1}{Z_n^{\alpha,\beta,\gamma}} \int_{[-1,1]^n} |\Delta(x)|^{2c}\,\mathrm{d}\mu_t(x_1)\cdots\mathrm{d}\mu_t(x_n) \tag{81}$$

where $\mathrm{d}\mu_t(x) = \prod_{j=1}^{r}(t_j - x)(1-x)^{a-1}(1+x)^{b-1}$, $Z_n^{\alpha,\beta,\gamma} = 2^{cn(n-1)+n(a+b+r-2)} S_n(a, b, c)$, $a = c(\alpha+1)$, $b = c(\beta+1)$, $c = \gamma + \frac{1}{2}$.

Hence, when $c = k$ is a positive integer, the symmetric Jacobi polynomials indexed by rectangular partitions are expressible as hyperdeterminants

$$Z_n^{\alpha,\beta,\gamma} p_{(n^r)}^{\alpha,\beta,\gamma}(t_1, \ldots, t_r) = n!\,D_n^{(k)}(c(t)) \tag{82}$$

where

$$c_m(t) = \int_{-1}^{1} x^m\,\mathrm{d}\mu_t(x). \tag{83}$$

This can be regarded as a generalization of the classical determinantal expression of the orthogonal polynomials in terms of the moments.

The extension of these identities to non-rectangular partitions or to other measures appears to be unknown.

5.2. The case $k = 1$: Leclerc's identity

On the other hand, for $k = 1$, Kaneko's representation can be extended to general orthogonal polynomials. Let μ be any linear functional such that the bilinear form $(f, g) = \mu(fg)$ is non-degenerate, and extend it as above to functions of n variables $x_1, \ldots, x_n$ by setting $\mu_n(f_1(x_1)\cdots f_n(x_n)) = \mu(f_1)\cdots\mu(f_n)$. Let $p_\lambda^{(k)}(x_1, \ldots, x_n)$ be the basis of symmetric polynomials obtained by applying the Gram–Schmidt process to the monomial basis with respect to the scalar product

$$\langle f, g\rangle_k = \mu_n(\Delta^{2k}(x) f(x) g(x)) \tag{84}$$

with leading term m_λ. In this section, we shall use the representation of partitions by *weakly increasing* sequences $\lambda = (0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_n)$ instead of the usual one (this will be more convenient for the indexing of minors). The one-variable polynomials $p_m(x)$ are the monic orthogonal polynomials associated with μ.

When $k = 1$, one has

$$p_\lambda^{(1)}(x) = \frac{D_\lambda(x)}{\Delta(x)} \tag{85}$$

where the alternants (Slater determinants) $D_\lambda(x) = \det\left(p_{\lambda_i+i-1}(x_j)\right)$ form the natural basis of antisymmetric orthogonal polynomials for μ_n. Now, we can write

$$R(y, x) = \prod_{j=1}^{r}\prod_{i=1}^{n}(y_j - x_i) = \frac{\Delta(x, y)}{\Delta(x)\Delta(y)}. \tag{86}$$

The analogue of Kaneko's integral in this context is the scalar product

$$\mu_n(\Delta^2(x) R(y, x)) = \langle 1, R(y, x)\rangle_1. \tag{87}$$

Expressing $\Delta(x, y)$ in terms of the one-variable monic orthogonal polynomials p_m as

$$\Delta(x, y) = \det(p_{i-1}(x_j) | p_{i-1}(y_k)) \tag{88}$$

and taking the Laplace expansion of this determinant of order $n + r$ with respect to its first n columns (containing the variables x_i), we find that

$$R(y, x) = \frac{1}{\Delta(x)\Delta(y)} \sum_{\alpha,\beta} (-1)^{|\alpha|} D_\alpha(x) D_\beta(y) = \sum_{\alpha,\beta} (-1)^{|\alpha|} p_\alpha^{(1)}(x) p_\beta^{(1)}(y) \tag{89}$$

where the sum runs over all pairs of partitions

$$\alpha = (0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_n) \qquad \beta = (0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_r) \tag{90}$$

such that $(\alpha_1 + 1, \ldots, \alpha_n + n, \beta_1 + 1, \ldots, \beta_r + r)$ is a permutation of $(1, 2, \ldots, n + r)$, in which case $(-1)^{|\alpha|}$ is its sign. In particular, for $\alpha = 0$, β is the rectangular partition $\beta = (n^r)$, so that

$$\begin{aligned}\langle 1, R(y, x)\rangle_1 &= \sum_{\alpha,\beta} (-1)^{|\alpha|} p_\beta^{(1)}(y) \left\langle p_0^{(1)}, p_\alpha^{(1)}\right\rangle \\ &= \mu_n(\Delta^2(x)) p_{(n^r)}^{(1)}(y_1, \ldots, y_r).\end{aligned} \tag{91}$$

This equation contains as a special case theorem 1 of [26], which in turn implies all the identities of Karlin and Szegö as well as many others. Indeed, taking the limit $x_i \to u$, $i = 1, \ldots, r$ in (85), we obtain a Wronskian of one-variable polynomials

$$p_\lambda^{(1)}(u, \ldots, u) = \frac{W\big(p_{\lambda_1}, p_{\lambda_2+1}, \ldots, p_{\lambda_r+r-1}\big)(u)}{1!2!\cdots(r-1)!} \tag{92}$$

(cf [29], 7.1.1 p 107), so that

$$\mu_n\left(\Delta^2(x)\prod_{i=1}^n (y-x_i)^r\right) = \mu_n(\Delta^2)\frac{W(p_n, \ldots, p_{n+r-1})(y)}{1!2!\cdots(r-1)!}. \tag{93}$$

But we have also

$$\mu_n\left(\Delta^2(x)\prod_{i=1}^n (y-x_i)^r\right) = (-1)^{nr} n! \det(c_{r+i+j}(y))|_0^{n-1} \tag{94}$$

where

$$c_m(y) = \mu[(x-y)^m] = \sum_{j=0}^m \binom{m}{j}\mu(x^j)(-y)^{m-j}. \tag{95}$$

The equality of the right-hand sides of (93) and (94) is precisely theorem 1 of [26].

Applying this identity (with $y = 0$) to the case where the moments c_n are the Bell polynomials $b_n(a) = \mu(x^n)$, so that $P_n(x) = C_n^{(a)}(x)$, we obtain (53).

This suggests the conjecture that in general $\mu_n(\Delta^{2k}(x)R(y,x))$ should be expressible as $\mu_n(\Delta^{2k}(x))q_{(n^r)}^{(k')}(y)$, where the $q_\lambda^{(k')}$ are the symmetric orthogonal polynomials for another functional μ' related to μ in some natural way.

5.3. *Hyperturánians of Legendre polynomials*

Let us start, as in [17], with the Legendre polynomials $P_n(x)$. Laplace's integral representation

$$P_n(x) = \frac{1}{\pi}\int_0^\pi \left(x + \cos\phi\sqrt{x^2-1}\right)^n \mathrm{d}\phi \tag{96}$$

can be rewritten as

$$P_n(x) = \int_a^b t^n \,\mathrm{d}\mu(t) \tag{97}$$

where $a = x - \sqrt{x^2-1}$, $b = x + \sqrt{x^2-1}$ and $\mathrm{d}\mu(t) = \pi^{-1}(t-a)^{-1/2}(b-t)^{-1/2}$. Hence,

$$D_n^{(k)}(P(x)) = \frac{1}{n!\pi^n}\int_{[a,b]^n} \Delta^{2k}(t)\prod_{i=1}^n (t_i-a)^{-1/2}(b-t_i)^{-1/2}\,\mathrm{d}t_i \tag{98}$$

which under the substitution $t_i = (b-a)u_i + a$ reduces to the Selberg integral

$$\begin{aligned} D_n^{(k)}(P(x)) &= \frac{1}{n!\pi^n}(b-a)^{kn(n-1)}\int_{[0,1]^n} \Delta^{2k}(u)\prod_{i=1}^n u_i^{-1/2}(1-u_i)^{-1/2}\,\mathrm{d}u_i \\ &= \frac{1}{n!\pi^n}\left(2\sqrt{x^2-1}\right)^{kn(n-1)} S_n\left(\frac{1}{2}, \frac{1}{2}; k\right). \end{aligned} \tag{99}$$

Now, the shifted polynomials $c_n = P_{r+n}(x)$ are the moments

$$P_{r+n}(x) = \int_a^b t^n \,\mathrm{d}\mu_r(t) \tag{100}$$

where $\mathrm{d}\mu_r(t) = \pi^{-1}t^r(t-a)^{-1/2}(b-t)^{-1/2}\,\mathrm{d}t$, so that

$$D_{n;r}^{(k)}(P(x)) = \frac{1}{n!\pi^n}\int_{[a,b]^n}\Delta^{2k}(t)(t_1\cdots t_n)^r\prod_{i=1}^n(t_i-a)^{-1/2}(b-t_i)^{-1/2}\,\mathrm{d}t_i$$
$$= \frac{(b-a)^{kn(n-1)+rn}}{n!\pi^n}\int_{[0,1]^n}\Delta^{2k}(u)\prod_{i=1}^n(u_i+v)^r u_i^{-1/2}(1-u_i)^{-1/2}\,\mathrm{d}u_i \qquad (101)$$

where $v = \frac{a}{b-a}$. This is of the form (67) with $y_1 = y_2 = \cdots = y_r = -v$, whence, since $1+2v = \frac{x}{\sqrt{x^2-1}}$,

$$D_{n;r}^{(k)}(P(x)) = 2^{kn(n-1)}(x^2-1)^{\frac{1}{2}(kn(n-1)+nr)}\frac{1}{n!\pi^n}S_n\left(\frac{1}{2},\frac{1}{2};k\right)$$
$$\times P_{(n^r)}^{\alpha,\beta,\gamma}\left(\frac{x}{\sqrt{x^2-1}},\ldots,\frac{x}{\sqrt{x^2-1}}\right) \qquad (102)$$

where $\alpha = \frac{1}{2k}-1$. $\beta = \frac{1}{2k}-1$ and $\gamma = k-\frac{1}{2}$.

It is instructive to have a look at the case $k=1$. Here, $\alpha=\beta=-\frac{1}{2}$ and the generalized Jacobi polynomials are the symmetric orthogonal polynomials for the measure

$$\mathrm{d}\mu(y) = \Delta^2(y)\prod_{i=1}^r\frac{\mathrm{d}y_i}{\sqrt{1-y_i^2}}. \qquad (103)$$

For $r = 1$, the orthogonal polynomials are the Chebyshev polynomials $T_n(y)$, and the symmetric orthogonal polynomials for (103) are the $D_\mu(y)/\Delta(y)$ formed from the corresponding monic polynomials t_m. Taking the limit of these expressions for $(y_1,\ldots,y_r)\to(\xi,\ldots,\xi)$, where $\xi = \frac{x}{\sqrt{x^2-1}}$, we obtain a Wronskian of Chebyshev polynomials evaluated at ξ, which is precisely the expression of the Turánian found by Karlin and Szegö (see also [26]).

5.4. Laguerre

We start with the hypergeometric representation of the monic Laguerre polynomials

$$\tilde{L}_n^{(a)}(x) = {}_1F_1\left(\left.\begin{matrix}-n\\a+1\end{matrix}\right|x\right) = \lim_{b\to\infty}{}_2F_1\left(\left.\begin{matrix}-n,b\\a+1\end{matrix}\right|\frac{x}{b}\right). \qquad (104)$$

The second part of this equality leads us to write each shifted hyperturánian as the limit of a Kaneko integral, which gives after simplification

$$D_{n;r}^{(k)}(\tilde{L}^{(a)}) = \frac{1}{n!k!^n}\lim_{b\to\infty}\left(\frac{-x}{b}\right)^{kn(n-1)+nr}\prod_{j=0}^{n-1}\frac{(b)_{jk+r}(a-b+1)_{jk}(jk+k)!}{(a+1)_{k(n+j-1)+r}}$$
$$\times P_{(n^r)}^{(\frac{b}{k}-1,\frac{a-b+1}{k}-1)}\left(\frac{b}{x},\ldots,\frac{b}{x};k\right). \qquad (105)$$

From (69), we see that this can be written as a generalized hypergeometric function

$$D_{n;r}^{(k)}(\tilde{L}^{(a)}) = (-1)^{\frac{kn(n-1)}{2}+nr}x^{kn(n-1)+nr}\frac{1}{n!k!^n}\prod_{j=0}^n\frac{(jk+k)!}{(a+1)_{k(n+j-1)+r}}$$
$$\times {}_2F_0^{(k)}\left(\left.\begin{matrix}-n,\frac{a+r}{k}+n-1\\-\end{matrix}\right|\frac{k}{x},\ldots,\frac{k}{x}\right) \qquad (106)$$

In particular, if $r=0$, we obtain

$$D_n^k(\tilde{L}^{(a)}) = (-1)^{\frac{kn(n-1)}{2}}x^{kn(n-1)}\frac{1}{n!k!^n}\prod_{j=0}^{n-1}\frac{(jk+k)!}{(a+1)_{k(n+j-1)}}. \qquad (107)$$

5.5. Hermite

We start as above with the representation of the monic Hermite polynomials as limits of hypergeometric functions

$$\tilde{H}_n(x) = \lim_{a\to\infty} a^{\frac{n}{2}}\,{}_2F_1\left(\begin{matrix}-n, 2a\\ a\end{matrix}\,\middle|\,\frac{1}{2}\left(1-\frac{x}{\sqrt{a}}\right)\right). \tag{108}$$

We can then write the shifted hyperturánian as the limit of a Kaneko integral.

If $r > 0$, from (79), one finds

$$D_{n,r}^{(k)}(\tilde{H}) = (-1)^{\frac{1}{2}kn(n-1)}2^{-\frac{1}{2}kn(n-1)-nr}k^{\frac{nr}{2}}\frac{1}{n!k!^n}\prod_{j=1}^{n}(jk)!H_{(n^r)}\left(\frac{x}{\sqrt{k}},\ldots,\frac{x}{\sqrt{k}};k\right). \tag{109}$$

In the simplest case $r = 0$, we obtain

$$D_n^{(k)}(\tilde{H}) = \left(-\frac{1}{2}\right)^{\frac{1}{2}kn(n-1)}\frac{1}{n!k!^n}\prod_{j=1}^{n}(jk)!. \tag{110}$$

Let us remark that this calculation is connected to the evaluation of $\left(\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}\right)^N \Delta(x)^{2k}$, $\left(N = \frac{kn(n-1)}{2}\right)$, which can be found in [30] (17.6.9). Indeed, expanding $\Delta(x)^{2k}$, one has

$$\left(\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}\right)^N \Delta(x)^{2k} = \sum_{\sigma_1,\ldots,\sigma_{2k}\in\mathfrak{S}_n}\epsilon(\sigma_1)\cdots\epsilon(\sigma_{2k})$$
$$\times\left(\sum_{l_1,\ldots,l_k}\binom{N}{l_1\cdots l_n}\prod_{j=1}^{n}\frac{\partial^{2l_j}}{\partial x_j^{2l_j}}\right)\prod_{i=1}^{n}x_i^{\sigma_1(i)+\cdots+\sigma_{2k}(i)-2k}. \tag{111}$$

But for each monomial $x_1^{p_1}\cdots x_n^{p_n}$ appearing in the previous formula, we obtain

$$\left(\sum_{l_1,\ldots,l_k}\binom{N}{l_1\cdots l_n}\prod_{j=1}^{n}\frac{\partial^{2l_j}}{\partial x_j^{2l_j}}\right)\prod_{i=1}^{n}x_i^{p_1}\Bigg|_{x_i=0} = \begin{cases}N!\prod_{i=1}^{n}\frac{p_i!}{\frac{p_i}{2}!} & \text{if each } p_i \text{ is even}\\ 0 & \text{otherwise.}\end{cases} \tag{112}$$

It follows that

$$\left(\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}\right)^N \Delta(x)^{2k} = (-1)^N n!2^{2N}N!D_n^{(k)}(\tilde{H}) \tag{113}$$

$$= \frac{2^N N!}{k!^n}\prod_{j=1}^{n}(jk)! \tag{114}$$

5.6. Charlier

The monic Charlier polynomials $C_n^{(a)}(x) = n!L_n^{(x-n)}(a)$ are given by the exponential generating function

$$\sum_{n\geqslant 0}C_n^{(a)}(x)\frac{t^n}{n!} = \mathrm{e}^{-at}(1+t)^x. \tag{115}$$

One has the integral representation (see [16] p 446)

$$C_n^{(a)}(x) = \frac{1}{\Gamma(-x)}\int_0^\infty \mathrm{e}^{-t}t^{-x-1}(t-a)^n\,\mathrm{d}t. \tag{116}$$

From this, we get easily the Hankel hyperdeterminants associated with the sequence $c_n = C_n^{(a)}(x)$. The result can be cast in the form

$$D_n^{(k)}(c) = \frac{1}{n!}\prod_{j=0}^{n-1}\frac{(k+kj)!\Gamma(-x+kj)}{k!\Gamma(-x)}. \tag{117}$$

In particular, for the determinants,

$$D_n^{(1)}(c) = \prod_{j=0}^{n-1}\frac{\Gamma(j+1)\Gamma(j-x)}{\Gamma(-x)}. \tag{118}$$

The shifted hyperdeterminants $D_{n;r}^{(k)}(c)$ can be similarly evaluated via Kaneko's identity in the Laguerre form,

$$D_{n;r}^{(k)}(c) = \frac{1}{n!k!^n}\prod_{j=0}^{n-1}(-x)_{jk+r}(jk+k)!L_{n^r}^{-\frac{x}{k}-1}\left(\frac{a}{k},\ldots,\frac{a}{k};k\right). \tag{119}$$

For the shifted determinants, we get a Wronskian of Laguerre polynomials which is easily seen to be equivalent to the evaluation given in [17].

5.7. Meixner

The Meixner polynomials admit the integral representation ([16] p 448)

$$\phi_n(-x;\beta,\gamma) = \frac{\Gamma(\beta)}{\Gamma(\beta-x)\Gamma(x)}\int_0^1 t^{x-1}(1-t)^{\beta-x-1}\left[1+\left(\frac{1}{\gamma}-1\right)t\right]^n \mathrm{d}t. \tag{120}$$

From this, one deduces the hyperdeterminants associated with $c_n = \phi(-x;\beta,\gamma)$

$$D_n^{(k)}(c) = \frac{1}{n!}\left(\frac{1-\gamma}{\gamma}\right)^{nk(n-1)}\prod_{j=0}^{n-1}\frac{\Gamma(x+jk)\Gamma(\beta-x+jk)\Gamma(\beta)\Gamma(jk+k+1)}{\Gamma(x)\Gamma(\beta-x)\Gamma(\beta+(n+j-1)k)\Gamma(k+1)}. \tag{121}$$

Kaneko's identity gives directly the shifted hyperdeterminants as

$$D_{n;k}^{(k)}(r) = \frac{1}{n!}\left(\frac{1-\gamma}{\gamma}\right)^{nk(n-1)+nr}\prod_{j=0}^{n-1}\frac{\Gamma(x+jk+r)\Gamma(\beta-x+jk)\Gamma(\beta)\Gamma(jk+k+1)}{\Gamma(x)\Gamma(\beta-x)\Gamma(\beta+r+(n+j-1)k)\Gamma(k+1)}$$
$$\times P_{(n^r)}^{(a,b)}(p,\ldots,p;k) \tag{122}$$

with $a = \frac{x}{k}-1$, $b = \frac{\beta-x}{k}-1$ and $p = \frac{\gamma}{\gamma-1}$.

5.8. Krawtchouk

The Krawtchouk polynomials are given by

$$K_n(x;p,N) = {}_2F_1\left(-n,-x;-N;\frac{1}{p}\right). \tag{123}$$

Assuming at first that $-N$ is not a negative integer, we can write down an integral representation

$$K_n(x;p,N) = \frac{\Gamma(-N)}{\Gamma(-x)\Gamma(-N+x)}\int_0^1 t^{-x-1}(1-t)^{-N+x-1}\left(1-\frac{t}{p}\right)^n \mathrm{d}t \tag{124}$$

which leads immediately to the evaluation

$$D_n^{(k)}(K) = \frac{1}{n!}\left(\frac{\Gamma(-N)}{\Gamma(-x)\Gamma(-N+x)}\right)^n\left(-\frac{1}{p}\right)^{kn(n-1)} S_n(-x,x-N,k). \tag{125}$$

After simplification, we find the expression

$$D_n^{(k)}(K) = \frac{1}{n!\,p^{kn(n-1)}} \prod_{j=0}^{n-1} \frac{(-x)_{jk}(N-x)_{jk}(jk+k)!}{(N)_{k(n+j-1)+r}k!} \tag{126}$$

which is well defined for integral N, provided that all the elements of the hyperdeterminant are also defined (recall that K_n is defined only for $n = 0, \ldots, N$).

The shifted hyperturánians can be evaluated from Kaneko's integral,

$$D_{n;r}^{(k)}(K) = \frac{(-1)^{nr}}{n!k!^n} \left(\frac{1}{p}\right)^{kn(n-1)+nr} \prod_{j=0}^{n-1} \frac{(-x)_{jk+r}(-N+x)_{jk}(kj+k)!}{(-N)_{k(n+j-1)+r}} P_{(n^r)}^{(\alpha,\beta)}(p, \ldots, p; k) \tag{127}$$

where $\alpha = -\frac{x}{k} - 1$ and $\beta = \frac{x-N}{k} - 1$. In particular, for $k = 1$, we obtain a Wronskian of Jacobi polynomials with parameters $-x - 1$ and $-N + x - 1$.

6. Hankel hyperdeterminants and symmetric functions

In this section, we shall give an expression of the Hankel hyperdeterminant $D_n^{(k)}(c)$ in terms of symmetric functions. Precisely, we suppose here that $c_n = h_n(x)$, the nth complete homogeneous symmetric function of some auxiliary set of variables $x = \{x_i\}$, and our aim is to obtain an expression of the symmetric function $D_n^{(k)}(h)$ in terms of the Schur functions $s_\lambda(x)$ (see [28] for notation). It turns out that this problem is equivalent to finding the Schur expansion of the even powers of the Vandermonde determinant, a difficult problem which has been thoroughly discussed in recent years, mainly in view of its potential applications to Laughlin's theory of the fractional quantum Hall effect (see [33] and references therein).

Since $D_n^{(k)}(h)$ is a homogeneous polynomial of degree n in the h_i, its Schur expansion will only involve partitions of length at most n. We can, therefore, assume that $x = \{x_1, \ldots, x_n\}$.

It will be convenient to work with Laurent polynomials in x. In particular, for each vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, we define the augmented monomial symmetric function

$$\tilde{m}_\lambda = \sum_{\sigma \in \mathfrak{S}_n} x^{\sigma\lambda}$$

where $\sigma(\lambda_1, \ldots, \lambda_n) = (\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)})$ and $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$.

Let ϕ be the linear map sending $\tilde{m}_\lambda$ to h_λ if $\lambda \in \mathbb{N}^n$ and 0 otherwise. As the set of the $\tilde{m}_\lambda$, with λ a decreasing sequence, is a basis of the space of symmetric Laurent polynomials the map ϕ is well defined.

Let us consider now the alternants

$$a_\lambda = \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) x^{\sigma\lambda}. \tag{128}$$

The image by ϕ of the product of $2k$ alternants is a hyperdeterminant. Since

$$\prod_{i=0}^{2k} a_{\lambda^{(i)}} = \sum_{\sigma_1,\ldots,\sigma_{2k-1} \in \mathfrak{S}_n} \tilde{m}_{\lambda^{(1)}+\sigma_1\lambda^{(2)}+\cdots+\sigma_{2k-1}\lambda^{(2k)}} \tag{129}$$

we get

$$\phi\left(\prod_{i=0}^{2k} a_{\lambda^{(i)}}\right) = \mathrm{Det}_{2k}\left(h_{\lambda_{i_1}^{(1)}+\cdots+\lambda_{i_{2k}}^{(2k)}}\right)\Big|_1^n. \tag{130}$$

The case where $k = 1$ is well known and can be found as an exercise in [28]. It is shown there that for any symmetric function f, we have

$$\phi(f a_\delta a_{-\delta}) = f \tag{131}$$

where $\delta = (n-1, \ldots, 2, 1, 0)$. In particular,

$$\begin{aligned} D_n^{(k)}(h) &= \mathrm{Det}_{2k}\left(h_{i_1+\cdots+i_{2k}}\right)\Big|_0^{n-1} = \phi\left(a_\delta^k\right) = \phi(\Delta(x)^{2k}) \\ &= \phi\left((-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^{2k} x_i^{n-1} \Delta(x)^{2(k-1)} a_\delta a_{-\delta}\right) \\ &= (-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} x_i^{n-1} \Delta(x)^{2(k-1)}. \end{aligned} \tag{132}$$

Since we are working with n variables, the effect of the factor $\prod_{i=1}^n x_i^{n-1}$ is to shift the parts of the partitions occurring in the Schur expansion of $\Delta(x)^{2(k-1)}$ by $n-1$.

The Schur expansions of Δ^2, which determine all Hankel hyperdeterminants of order 4, have been computed up to nine variables in [33] and recently up to ten variables by Wybourne. Using the Littlewood–Richardson rule, it is then possible to compute the powers Δ^{2k} for small values of k. The first cases are

$$\begin{aligned} D_2^{(2)}(h) &= -s_{31} + 3s_{22} \\ D_3^{(2)}(h) &= -s_{642} + 3s_{633} + 3s_{552} - 6s_{543} + 15s_{444} \\ D_2^{(3)}(h) &= -s_{51} + 5s_{42} - 10s_{33} \\ D_3^{(3)}(h) &= -s_{1062} + 5s_{1053} - 10s_{1044} + 5s_{972} - 20s_{963} + 25s_{954} \\ &\quad - 10s_{882} + 25s_{873} + 15s_{864} - 100s_{855} - 100s_{774} + 160s_{765} - 280s_{666} \\ D_2^{(4)}(h) &= -s_{71} + 7s_{62} - 21s_{53} + 35s_{44}. \end{aligned}$$

In terms of the elementary symmetric functions e_n and the power sums p_n, this identity can be rewritten as

$$D_n^{(k)}(h) = (-1)^{n(n-1)/2} e_n^{n-k} \det(p_{n-i+j})^{k-1}. \tag{133}$$

As an illustration of (132), let us consider the case where $c_n = U_n(x)$, the nth Chebyshev polynomial of the second kind. From the generating function

$$\sum_{n\geqslant 0} U_n(x) t^n = \frac{1}{1 - 2xt + t^2}$$

we see that $U_n(x) = h_n(x_1, x_2)$, where $x_1 + x_2 = 2x$ and $x_1 x_2 = 1$. Hence, $\Delta^2 = (x_1 + x_2)^2 - 4x_1x_2 = 4(x^2 - 1)$, so that $D_2^{(k)}(U) = -[4(x^2-1)]^{(k-1)}$. Comparing with (4), we obtain the identity

$$\frac{1}{2}\sum_{j=0}^{m} (-1)^j \binom{m}{j} U_j(x) U_{m-j}(x) = \begin{cases} [4(1-x^2)]^{\frac{m}{2}-1} & m \text{ even} \\ 0 & \text{otherwise.} \end{cases} \tag{134}$$

Specializing to the Fibonacci numbers $f_n = (-\mathrm{i})^n U_n(\mathrm{i}/2)$, (normalized such that $f_0 = f_1 = 1$), we find

$$\frac{1}{2}\sum_{j=0}^{2k} (-1)^j \binom{2k}{j} f_j f_{2k-j} = 5^{k-1}. \tag{135}$$

6.1. An application: inverse factorials

The symmetric function approach allows us to handle the case

$$c_n = \frac{1}{n!}. \tag{136}$$

Indeed, the image of a Schur function s_λ under the specialization $h_n \mapsto c_n$ is equal to the scalar product

$$\frac{1}{N!}\langle s_\lambda, s_1^N\rangle \tag{137}$$

where $N = |\lambda|$. If λ has at most n parts, we can interpret the above as the scalar product of rational $GL(n)$-characters, defined by

$$\langle f(x), g(x)\rangle_{GL(n)} = \frac{1}{n!}\mathrm{CT}\left\{ f(x)\prod_{i\neq j}\left(1-\frac{x_i}{x_j}\right) g(\bar{x})\right\} \tag{138}$$

where CT means the constant term, and $\bar{x} = \left(x_1^{-1}, \ldots, x_n^{-1}\right)$.

Hence, we can write

$$\begin{aligned}
\mathrm{I}(k,n) = D_n^{(k)}(c) &= \frac{1}{[kn(n-1)]!}\left\langle (-1)^{\frac{n(n-1)}{2}}(x_1\cdots x_n)^{n-1}\Delta^{2k-2}(x), (x_1+\cdots+x_n)^{kn(n-1)}\right\rangle \\
&= \frac{(-1)^{\frac{kn(n-1)}{2}}}{[kn(n-1)]!}\frac{1}{n!}\mathrm{CT}\left\{(x_1\cdots x_n)^{k(n-1)}\prod_{i\neq j}\left(1-\frac{x_i}{x_j}\right)^k\right. \\
&\quad \left. \times\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)^{kn(n-1)}\right\} \\
&= \frac{(-1)^{\frac{kn(n-1)}{2}}}{[kn(n-1)]!}\langle (x_1\cdots x_n)^{k(n-1)}, (x_1+\cdots+x_n)^{kn(n-1)}\rangle'_\alpha
\end{aligned} \tag{139}$$

where in the last equation we have introduced Macdonald's second scalar product $\langle\ ,\ \rangle'_\alpha$ associated with Jack polynomials in n variables, with parameter $\alpha = \frac{1}{k}$ (see [28] (10.35) p 183).

Now, $(x_1\cdots x_n)^{k(n-1)} = P^{(\alpha)}_{(k(n-1))^n}$, and

$$(x_1+\cdots+x_n)^{kn(n-1)} = \sum_{\kappa\vdash kn(n-1)} C_\kappa^{(\alpha)}(x) \tag{140}$$

where, if $\kappa \vdash N$,

$$C_\kappa^{(\alpha)}(x) = \frac{\alpha^N N!}{c_\kappa(\alpha)} Q_\kappa^{(\alpha)}(x) \tag{141}$$

with

$$c_\kappa(\alpha) = \prod_{(i,j)\in\kappa}(\alpha(\kappa_i - j) + \kappa'_j - i + 1). \tag{142}$$

Therefore, if we denote by ν the rectangular partition $(k(n-1))^n$ of weight $N = kn(n-1)$, the Hankel hyperdeterminant is given by

$$\mathrm{I}(k,n) = \frac{(-1)^N}{k^{N)}c_\nu(k^{-1})}\left\langle P_\nu^{(1/k)}, Q_\nu^{(1/k)}\right\rangle'_{1/k} \tag{143}$$

which can be evaluated in closed form thanks to equation (10.37) of [28] p 183. This yields

$$\mathrm{I}(k,n) = \frac{(-1)^{kn(n-1)/2}(kn)!}{n!(k!)^n}\prod_{i=0}^{n-1}\frac{(ki)!}{(k(n+i-1))!}. \tag{144}$$

The case $k = 1$ could have been obtained in a simpler way from the hook-length formula giving the dimensions of the irreducible representations of the symmetric group. Actually, (144) can also be derived from Selberg's formula (see equation (A.9)).

7. Conclusion

We have demonstrated that the calculation of Hankel hyperdeterminants amounts to the evaluation of an interesting class of multidimensional integrals, including Selberg's and Kaneko's ones, and more generally, the partition functions of one-dimensional Coulomb systems with logarithmic potential. We have presented a series of examples which can be evaluated more or less directly from known results, and obtained in our way a unified presentation of many Hankel determinants, including some new cases. However, obtaining new integrals from algebraic or combinatorial evaluation of hyperdeterminants would be more interesting. A few examples are presented in this paper, and we expect more from a systematic study of Hankel hyperdeterminants from an invariant theory point of view. Indeed, any generalization of one of the various tricks working with ordinary Hankel determinants would immediately lead to new interesting integrals.

Appendix A. Hyperdeterminantal aspects of Selberg's original proof

The Selberg integral can be deduced from the Hilbert, factorial or inverse factorial hyperdeterminants. Actually, the evaluation of any non-trivial class of Hankel hyperdeterminants would lead either to Selberg's integral in full generality, or to some interesting generalization. This is already apparent in Selberg's original proof. The first part of his proof can be translated in terms of hyperdeterminants in the following way. First, we write Selberg's integral, for $c = k$ an integer, as a hyperdeterminant

$$S_n(a,b,k) = n!B(\alpha,\beta)^n D_n^{(k)}(c(\alpha,\alpha+\beta)) \tag{A.1}$$

where $c_n(a,b) = \frac{(a)_n}{(b)_n}$ and $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol. Computing $S_n(a,b,k)$ amounts to finding a closed form for $D_n^{(k)}(c(a,b))$.

Following Selberg's proof, we can write our hyperdeterminant as the sum

$$D_n^{(k)}(c(a,b)) = \sum_J c_J \prod_{i=1}^{n}\frac{(a)_{j_l}}{(b)_{j_l}} \tag{A.2}$$

where J runs over the integer vectors $J = (j_1,\ldots,j_n)$ such that $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_n$, $j_1 + \cdots + j_n = kn(n-1)$, and $c_J \in \mathbb{Z}$. Let us remark that this is a general fact. For any Hankel tensor $\left(h_{i_1+\cdots+i_{2k}}\right)_{1\leqslant i_1,\ldots,i_{2k}\leqslant n}$, the hyperdeterminant can be written as

$$D_n^{(k)}(h) = \sum_{j_1\leqslant\cdots\leqslant j_n}\left(\sum_{\substack{\sigma_1,\ldots,\sigma_{2k}\in\mathfrak{S}_n\\ \sigma_1(1)+\cdots+\sigma_{2k}(1)=j_1\\ \cdots\sigma_1(n)+\cdots+\sigma_{2k}(n)=j_n}}\epsilon(\sigma_1)\cdots\epsilon(\sigma_n)\right)h_{j_1}\cdots h_{j_n}. \tag{A.3}$$

One can see that the sum of the j_i is necessarily equal to $kn(n-1)$.

Now, the right-hand side of (A.2) can be rewritten as

$$D_n^{(k)}(c(a,b)) = \left(\prod_{m=1}^{n} \frac{(a)_{k(m-1)}}{(b)_{k(n+m-2)}}\right)\left(\sum_J c_J \prod_{m=1}^{n} \frac{(a)_{j_m}(b)_{k(n+m-2)}}{(a)_{k(m-1)}(b)_{j_m}}\right). \qquad \text{(A.4)}$$

A straightforward investigation of the $2k$-uplets of permutations giving a nonzero c_J implies that for each $m \in \{1, \ldots, n\}$, one has

$$k(m-1) \leqslant j_m \leqslant k(n+m-2). \qquad \text{(A.5)}$$

Hence, our hyperdeterminant can be written as a product

$$D_n^{(k)}(c(a,b)) = \prod_{m=1}^{n} \frac{(a)_{k(m-1)}(b-a)_{k(m-1)}}{(b)_{k(n+m-2)}} \times \frac{P(a, b-a)}{Q(b-a)} \qquad \text{(A.6)}$$

where $P(a,b)$ is a polynomial whose degree in b is at most $\frac{kn(n-1)}{2}$ and $Q(b)$ is a polynomial of degree $\frac{kn(n-1)}{2}$.

Since $D_n^{(k)}(c(a,b))$ is symmetric in a and $b-a$, we see that the ratio $\frac{P(a,b-a)}{Q(b-a)} = \alpha(n,k)$ is independent of a and b, so that

$$D_n^{(k)}(c(a,b)) = \alpha(n,k) \prod_{m=1}^{n} \frac{(a)_{k(m-1)}(b-a)_{k(m-1)}}{(b)_{k(n+m-2)}} \qquad \text{(A.7)}$$

Now, setting $a = 1$ and $b = 2$ we obtain

$$D_n^{(k)}(c(a,b)) = \prod_{m=1}^{n} \frac{(1+k(m+n-2))!(a)_{k(m-1)}(b-a)_{k(m-1)}}{(k(m-1))!^2(b)_{k(m+n-2)}} \mathrm{H}(k,n) \qquad \text{(A.8)}$$

that is, Selberg's integral can be deduced in full generality from the Hilbert hyperdeterminant.

Also, observing that the case of inverse factorials is given by the limit

$$\mathrm{I}(k,n) = \lim_{L\to\infty} L^{-kn(n-1)} D_n^{(k)}(c(L+1,1)) \qquad \text{(A.9)}$$

we have

$$D_n^{(k)}(c(a,b)) = (-1)^{\frac{kn(n-1)}{2}} \prod_{m=1}^{n}\prod_{m=1}^{n} \frac{(a)_{k(m-1)}(b-a)_{k(m-1)}}{(k(n+m-2))!(b)_{k(n+m-2)}} \mathrm{I}(k,n). \qquad \text{(A.10)}$$

In the same way, the Hankel hyperdeterminant of factorial numbers

$$\mathrm{F}(n,k) = \lim_{L\to\infty} L^{kn(n-1)} D_n^{(k)}(c(1,L+1)) \qquad \text{(A.11)}$$

gives another equivalent identity

$$D_n^{(k)}(c(a,b)) = \prod_{m=1}^{n} \frac{(a)_{k(m-1)}(b-a)_{k(m-1)}}{(k(m-1))!(b)_{k(n+m-2)}} \mathrm{F}(n,k). \qquad \text{(A.12)}$$

Appendix B. Possible generalizations

More generally, if we start with a hypergeometric moment sequence $c_n = \frac{P(n)}{Q(n)} c_{n-1}$, where $P(n) = \sum_{i=0}^{r} a_i n^i$ and $Q(n) = \sum_{i=0}^{s} b_i n^i$ are two polynomials in n, a similar analysis leads to an expression of the hyperdeterminant $D_n^{(k)}(c)$ as a product

$$D_n^{(k)}(c) = c_0^n \prod_{m=0}^{n-1} \frac{\prod_{j=0}^{km-1} P(j)}{\prod_{j=0}^{k(n+m-1)-1} Q(j)} R_n^{(k)}(\underline{a}; \underline{b}) \qquad \text{(B.1)}$$

where $R_n^{(k)}(\underline{a}; \underline{b})$ is a polynomial of degree at most $\frac{kn(n-1)}{2}$ in both sets of variables $\underline{a} = \{a_0, \ldots, a_r\}$ and $\underline{b} = \{b_0, \ldots, b_s\}$ and whose coefficients are in $\mathbb{Z}$.

In the most general case, $R_n^{(k)}(\underline{a}; \underline{b})$ cannot be factorized, and seems difficult to compute. However, in some simple cases, we can give a closed form.

Suppose that $P(n) = cn^2 + bn + a$ and $Q(n) = 1$, we find

$$R_2^{(k)}(a, b, c; 1) = \frac{(2k+1)!}{k!} \prod_{i=k+2}^{2k+1} (b + cj). \tag{B.2}$$

Let us give now two examples involving combinatorial numbers.

The tri-Catalan numbers $C_n^{(3)} = \frac{\binom{3n}{n}}{2n+1}$ admit a representation as moments [32]

$$C_n^{(3)} = \frac{3^{3n+1}}{2\sqrt{3}\pi} \frac{B\left(n + \frac{1}{3}, n + \frac{2}{3}\right)}{2n+1} = \int_0^{\frac{27}{4}} x^n \, \mathrm{d}\mu(x) \tag{B.3}$$

where $\mathrm{d}\mu(x) = \frac{\sqrt{3} 2^{\frac{2}{3}}}{12\pi} \frac{2^{\frac{1}{3}} (27+3\sqrt{81-12x})^{\frac{2}{3}} - 6x^{\frac{1}{3}}}{x^{\frac{2}{3}} (27+3\sqrt{81-12x})^{\frac{1}{3}}} \mathrm{d}x$. It follows that our hyperdeterminant has the integral representation

$$D_n^{(k)}(C^{(3)}) = \frac{1}{n!} \int_0^{\frac{27}{4}} \cdots \int_0^{\frac{27}{4}} \Delta(x)^{2k} \prod_{i=1}^{n} \mathrm{d}\mu(x_i) \tag{B.4}$$

which looks rather difficult to compute. Nevertheless, our previous remarks allow us to start the calculation

$$D_n^{(k)}(C^{(3)}) = \prod_{m=0}^{n} \frac{\prod_{j=0}^{km-1} (-2 + 11j - 18j^2 + 9j^3)}{\prod_{j=0}^{k(n+m-1)-1} (4j^2 - j)} R_n^{(k)}(-2, 11, -18, 9; 0, -1, 4) \tag{B.5}$$

and it remains to find a closed form for $R_n^{(k)}(a_1, a_2, a_3, a_4; b_1, b_2, b_3)$. When $k = 1$, the result is known [39].

If $c_n = (2n)!$ we have

$$D_n^{(k)}(c) = \frac{1}{2^n} \int_0^{\infty} \cdots \int_0^{\infty} \Delta(x)^{2k} \prod_{i=1}^{n} \frac{\exp(-\sqrt{x_i})}{\sqrt{x_i}} \, \mathrm{d}x_i \tag{B.6}$$

$$= 2^{kn(n-1)} \prod_{m=0}^{n} (mk-1)! \prod_{j=0}^{km-1} (2j-1) R_n^{(k)}(0, -2, 4; 0). \tag{B.7}$$

Let us give now some polynomials $R_n^{(k)}(\underline{a}, \underline{b})$ for various values of n, k, $\underline{a}$ and $\underline{b}$:

$$R_2^{(1)}(a_0, a_1, a_2; b_0, b_1) = -a_0 b_1 + a_1 b_0 + 3a_2 b_0 + 2a_2 b_1$$

$$\begin{aligned} R_2^{(2)}(a_0, a_1, a_2; b_0, b_1) = 6\big(&-2a_0a_1b_0b_1 - 10a_0a_2b_0b_1 + 15a_1a_2b_0b_1 - a_0a_1b_1 - 15a_0a_2b_1^2 \\ &+ 9a_1a_2b_0^2 + a_0^2b_1^2 + a_1^2b_0^2 + 20a_2^2b_0^2 + 24a_2^2b_1^2 + a_1^2b_0b_1 + 50a_2^2b_0b_1\big) \end{aligned}$$

$$R_2^{(2)}(a_0, a_1, a_2, a_3;) = 6\big(a_1^2 + 9a_1a_2 + 20a_2^2 + a_0a_3 + 35a_1a_3 + 150a_2a_3 + 274a_3^2\big)$$

$$\begin{aligned} R_3^{(2)}(a_0, a_1, a_2;) = 16\big(&94\,251a_0a_1^3a_2^2 + 5525a_0^2a_1^2a_2^2 + 48a_0^3a_1a_2^2 + 1853\,066a_0a_1a_2^4 \\ &+ 603\,101a_0a_1^2a_2^3 + 25\,518a_0^2a_1a_2^3 + 7123a_0a_1^4a_2 + 522a_0^2a_1^3a_2 + 3278\,390a_1^3a_2^3 \\ &+ 15\,303\,958a_1^2a_2^4 + 384a_0^3a_2^3 + 41\,544a_0^2a_2^4 + 211a_0a_1^5 + 19a_0^2a_1^4 + 592a_1^6 \\ &+ 37\,115\,136a_2^6 + 2178\,696a_2^5a_0 + 3727\,7876a_2^5a_1 + 385\,834a_1^4a_2^2 + 23\,654a_1^5a_2\big) \end{aligned}$$

$$R_2^{(2)}(; b_0, b_1, b_2, b_3) = -6\big(b_0b_3 - b_1^2 - 11b_1b_2 - 45b_1b_3 - 30b_2^2 - 250b_2b_3 - 524b_3^2\big).$$

Appendix C. Pseudo-hyperdeterminants

For tensors of odd order, another notion of hyperdeterminant is considered for example in [36]

$$\mathrm{Det}_+\left(A_{i_1\cdots i_{2k+1}}\right)\Big|_0^{n-1} = \sum_{\sigma_1\cdots\sigma_{2k}} \epsilon(\sigma_1\cdots\sigma_{2k}) \prod_{i=1}^{n} A_{i\sigma_1(i)\cdots\sigma_{2k}(i)}. \tag{C.1}$$

Note that this polynomial does not have the same invariance properties as the hyperdeterminant under linear transformations.

When $A_{i_1\cdots i_{2k+1}} = c_{m_{i_1}+i_2+\cdots+i_{2k+1}}$, the c_n being the moments of a measure $\mathrm{d}\mu(x)$, this hyperdeterminant can be expressed as a multiple integral involving an even power of the Vandermonde determinant

$$\begin{aligned}
{}_+D_{\underline{m}}^{(k)}(c) &= \mathrm{Det}_+\left(c_{m_{i_1}+i_2+\cdots+i_{2k}}\right)_0^{n-1} \\
&= \sum_{\sigma_1\cdots\sigma_{2k}} \epsilon(\sigma_1\cdots\sigma_{2k}) \prod_{i=1}^{n} \int_a^b x^{m_i+\sigma_1(i)+\cdots\sigma_{2k}(i)-2k}\,\mathrm{d}\mu(x) \\
&= \int_a^b \cdots \int_a^b \sum_{\sigma_1\cdots\sigma_{2k}} \epsilon(\sigma_1\cdots\sigma_{2k}) \prod_{i=1}^{n} x_i^{m_i+\sigma_1(i)+\cdots\sigma_{2k}(i)-2k}\,\mathrm{d}\mu(x_i) \\
&= \int_a^b \cdots \int_a^b \prod_{i=1}^{n} x_i^{m_i}\,\Delta(x)^{2k}\,\mathrm{d}\mu(x_1)\cdots\mathrm{d}\mu(x_n).
\end{aligned} \tag{C.2}$$

Obviously, one has

$${}_+D_{(0^n)}^{(k)}(c) = n!D_n^{(k)}(c) \tag{C.3}$$

and we can compute other examples related to Selberg's integral. The main tool is the system of differential equations verified by the functions $f_{\underline{m}} = \Delta(x)^{2k}\prod_{i=1}^{n} x_i^{a+m_i-1}(1-x_i)^{b-1}$ (see Aomoto's proof of a variant of the Selberg integral in [1] or [30] for example).

Let us list a few results:

(i) For $c_n = \frac{1}{n+1}$, one has

$$\begin{aligned}
{}_+D_{(1^s,0^{n-s})}^{(k)} &= \int_0^1 \cdots \int_0^1 x_1\cdots x_s\,\Delta(x)^{2k}\,\mathrm{d}x_1\cdots\mathrm{d}x_n \\
&= \frac{1}{k!^n}\prod_{j=1}^{s}\frac{1+(n-j)k}{2+(2n-j-1)k}\prod_{j=0}^{n-1}\frac{(k(1+j))!(kj)!^2}{(1+(n+j-1)k)!}.
\end{aligned} \tag{C.4}$$

(ii) More generally, if $c_n = \frac{\Gamma(a+n)}{\Gamma(b+n)}$, one gets

$$\begin{aligned}
{}_+D_{(1^s,0^{n-s})}^{(k)}(c) &= \frac{1}{\Gamma(b-a)^n k!^n}\prod_{j=1}^{s}\frac{a+k(n-j)}{b+(2n-j-1)k} \\
&\quad\times\prod_{j=0}^{n-1}\frac{(k(1+j))!\Gamma(a+jk)\Gamma(b-a+jk)}{\Gamma(b+(n+j-1)k)}.
\end{aligned} \tag{C.5}$$

(iii) If $c_n = n!$, one obtains

$${}_+D_{(1^s,0^{n-s})}^{(k)}(c) = \frac{1}{k!^n}\prod_{j=1}^{m}(1+k(n-j))\prod_{j=0}^{n-1}(k(1+j))!(kj)! \tag{C.6}$$

and

$$
{}_+D^{(k)}_{(2^m,1^s,0^{n-m-s})}(c) = \frac{1}{k!^n}\prod_{j=1}^{m}(2+k(2n-m-j))\prod_{j=1}^{m+s}(1+k(n+j))\prod_{j=0}^{n-1}(k(1+j))!(kj)!. \tag{C.7}
$$

References

[1] Aomoto K 1987 Jacobi polynomials associated with Selberg integrals *SIAM J. Math. Anal.* **18** 545–9

[2] Baker T H and Forrester P J 1997 The Calogero–Sutherland model and generalized classical polynomials *Commun. Math. Phys.* **188** 175–216

[3] Barsky D and Carpentier M 1996 Polynômes de Jacobi généralisés et intégrales de Selberg *Electron. J. Comb.* **3** R1

[4] Barvinok A I 1995 New algorithms for linear k-matroid intersection and matroid k-parity problem *Math. Program.* **69** 449–70

[5] Cayley A 1843 On the theory of determinants *Trans. Camb. Phil. Soc.* **8** 1–16

[6] Cayley A 1846 Mémoire sur les hyperdéterminants *J. Reine Angew. Math.* **30** 1–37

[7] Cayley A 1851 On the theory of permutants *Camb. Dublin Math. J.* **8** 40–51

[8] Debiard A 1988 Système différentiel hypergéométrique et parties radiales des opérateurs invariants des espaces symétriques de type BC_p *Séminaire d'algèbre (Lecture Notes in Math. vol 1293)* ed M-P Malliavin (Berlin: Springer) pp 42–124

[9] Desainte-Catherine M and Viennot X G 1986 Enumeration of certain Young tableaux with bounded height *Combinatoire énumérative (Montréal 1985) (Lecture Notes in Mathematics vol 1234)* (Berlin: Springer) pp 58–67

[10] Forrester P J 1998 Random matrices, log-gases and the Calogero–Sutherland model *Quantum Many-Body Problems and Representation Theory (MSJ Mem. vol 1)* (Tokyo: Mathematical Society of Japan) pp 97–181

[11] Gelfand I M, Kapranov M M and Zelevinsky A V 1994 *Discriminants, Resultants and Multidimensional Determinants* (Boston, MA: Birkhäuser) p 523

[12] Gherardelli F 1993 Osservazioni sugli iperdeterminant *Istit. Lombardo Accad. Sci. Lett. Rend.* A **127** 107–13

[13] Haukkanen P 1992 Higher-dimensional GCD matrices *Linear Algebr. Appl.* **170** 53–63

[14] Heine H E 1878 *Handbuch der Kugelfunktionen* 2nd edn vol 1 ed G Reimer (Berlin)
Heine H E 1881 *Handbuch der Kugelfunktionen* 2nd edn vol 2 ed G Reimer (Berlin)

[15] Kaneko J 1993 Selberg integrals and hypergeometric fuctions associated with Jack polynomials *SIAM J. Math. Anal.* **24** 1086–110

[16] Karlin S 1968 *Total Positivity* (Stanford, CA: Stanford University Press)

[17] Karlin S and Szegö G 1961 On certain determinants whose elements are orthogonal polynomials *J. Anal. Math.* **8** 1–157

[18] Koekoek R and Swarttouw R F 1998 The Askey-scheme of hypergeometric polynomials and its q-analogue *Report* no 98-17 Department of Technical Mathematics and Informatics, Delft University of Technology

[19] Korányi A 1991 Hua-type integrals, hypergeometric functions and symmetric polynomials *Int. Symp. on Number Theory and Analysis in Memory of Hua Loo Keng Volume II: Analysis* ed Gong Sheng *et al* (Berlin: Springer) pp 169–80

[20] Krattenthaler C 1998 Advanced determinant calculus *The Andrews Festschrift (Séminaire Lotharingien de Combinatoire vol 42)* 67 pp (electronic)

[21] Kung J S and Rota G C 1984 The invariant theory of binary forms *Bull. Am. Math. Soc.* **10** 27–85

[22] Lassalle M 1991 Polynômes de Jacobi généralisés *C. R. Acad. Sci., Paris* I **312** 425–8

[23] Lassalle M 1991 Polynômes de Laguerre généralisés *C. R. Acad. Sci., Paris* I **312** 725–8

[24] Lassalle M 1991 Polynômes de Hermite généraliés *C. R. Acad. Sci., Paris* I **313** 579–82

[25] Lawden D F 1951 The function $\sum_{n=1}^{\infty} n^r z^n$ and associated polynomials *Proc. Camb. Phil. Soc.* **47** 309–14

[26] Leclerc B 1998 On certain formulas of Karlin and Szegö *Séminaire Lotharingien de Combinatoire* **41** electronic pp 21

[27] Luque J-G and Thibon J-Y 2002 Pfaffian and Hafnian identities in shuffle algebras *Adv. Appl. Math.* **29** 620–46

[28] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials* 2nd edn (Oxford: Clarendon)

[29] Mehta M L 1989 *Matrix Theory* (Hindustan Pub. Corp. and Les Éditions de Physique)

[30] Mehta M L 1991 *Random Matrices* 2nd edn (London: Academic)

[31] Penson K A and Sixdeniers J-M 2001 Integral representations of Catalan and related numbers *J. Integer Sequences* **4** Art. 01.2.5 (electronic)

[32] Penson K A and Solomon A I 2001 Coherent states from combinatorial sequences *Preprint* quant-ph/0111151

[33] Scharf T, Thibon J-Y and Wybourne B G 1994 Powers of the Vandermonde determinant and the quantum Hall effect *J. Phys. A: Math. Gen.* **27** 4211–9

[34] Selberg A 1944 Bemerkninger om et multiplet integral *Normat.* **26** 71–78

[35] Sloane N J *The On-Line Encyclopedia of Integer Sequences* available online at http://www.research.att.com/~njas/sequences/

[36] Sokolov N P 1960 Spatial matrices and their applications *Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow* (in Russian)

[37] Sokolov N P 1972 *Introduction to the Theory of Multidimensional Matrices* (Kiev: Nukova Dumka)

[38] Radoux C 1979 Calcul effectif de certains déterminants de Hankel *Bull. Soc. Math. Belg. Sér.* B **31** 49–55

[39] Tamm U 2001 Some aspects of Hankel matrices in coding theory and combinatorics *Electron. J. Comb.* **8** A1

[40] Vein R and Dale P 1999 *Determinants and their Applications in Mathematical Physics* (Berlin: Springer)

[41] Vretare L 1984 Formulas for elementary spherical functions and generalized Jacobi polynomials *SIAM J. Math. Anal.* **15** 805–33

[42] Vilenkin N Ja and Klimyk A U 1995 Representation of Lie groups and special functions. Recent advances *Mathematics and its Applications* vol 316 (Dordrecht: Kluwer)